MALTA REVIEW OF EDUCATIONAL RESEARCH

Volume 18, No. 2, 440-474



# Teaching for conceptual understanding: A mathematics lesson study

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**Abstract:** This article zooms in on a lesson study process which sees a group of mathematics teachers designing a research lesson to address the students' prevalent reliance on rote memorisation which is overshadowing the importance of conceptual understanding. Breaking away from the traditional approach of posing a textbook question, the team chose to present a Year 10 high-ability class with an open-ended investigation, allowing free exploration of a hexagonal visual aid. To ensure a structured and guided learning process, the lesson was designed on the 5E Inquiry-Based Instructional model (Bybee & Landes, 1990), a framework that was introduced to students as well. Through this self-directed learning opportunity, students took on roles as active agents fuelling a collaborative inquiry and using their existing knowledge as a scaffold for constructing new knowledge independently. The observations of this research lesson proved to be an enriching experience for everyone involved as the shared feedback revealed various successful aspects of the lesson. This experience motivated the lesson study team to intensify their efforts towards integrating lessons promoting students' conceptual understanding. Concurrently, the observations also shed light on the challenges of teaching for conceptual understanding, urging the lesson study team to reflect on their practices for ongoing improvement.

**Keywords:** Lesson study; mathematical connections; open-ended investigation; professional development

## Introduction

Stemming from Japan, lesson study is a form of professional development (PD) that distinguishes itself from other models mainly through its collaborative, and inquiry-based nature. Murata (2011) defines lesson study as when "teachers organically come together with a shared question regarding their pupils' learning, plan a lesson to make pupil learning visible, and examine and discuss what they observe" (p.2). In contrast with conventional PD models that often lack continuity, lesson study embraces an ongoing, iterative cycle of planning, observation, and reflection, immersing teachers in a continuous process of professional growth. It deeply values the expertise and voices of teachers, empowering them to delve into their own classrooms, identify areas of improvement and formulate questions that resonate with their specific contexts.

This question-driven approach, as Fuji (2016) accentuates, personalizes the process and transforms it into a research-oriented endeavour, enabling teachers to develop tailored solutions that address their unique classroom challenges. By becoming drivers of their own PD and engaging in their own inquiry, teachers acquire a deeper understanding of their practices and gain the agency to respond to the ever-changing needs of the students. The sense of ownership instilled, inspires teachers to "assert a social constructivist perspective" (Rock & Wilson, 2005, p.80), recognising that a lesson is not a mechanical process, but an organic system (Fuji, 2016).

Despite having participated in previous lesson studies, for the first author, stepping into a leadership position during this lesson study was both an exciting and challenging experience. Her leading role comprised proactive planning of ways to mitigate any apprehension and uncertainty that might stem from the lesson study team members' diverse working styles and pedagogical approaches. Throughout the process, her efforts were also geared towards providing ample space for an open dialogue, persuading teachers to take ownership of the process, encouraging them to express their ideas and queries and actively listening to the shared constructive feedback. Furthermore, to promote a sense of accountability and ensure active engagement, she advocated for flexibility in roles, allowing the team members to switch responsibilities during different phases of the lesson study.

Meanwhile, this report is written for fellow teachers, educators and leaders not to be disheartened by the rigorous process or discouraged by the challenges that may arise. Instead, it aims to provide valuable insights that motivate, assist and support educators in taking the leap and embark on a reflective, collaborative PD that has the potential to transform teaching practices and enhance student learning (Stepanek et al., 2006). The objective is not to present an "exemplary lesson plan" (Stepanek et al., 2006, p. 1) or an evaluation of the lesson study team members (Stigler & Hiebert, 1999, p. 1). Our intention is to share this practice so that we can inspire others to find time to pause, think and contemplate on the benefits of using lesson study to address issues with student learning. In our case we report on a lesson study process, led by the first author with a group of mathematics teachers at her school and supported through lectures and meetings with the second author at university, that explored the use of an open-ended mathematical investigation for student engagement and conceptual understanding.

**The research lesson**

It is not only a fact that a considerable number of students in schools are failing to meet the educational benchmarks (OECD, 2019), but the sad reality is that we are also witnessing a sense of disengagement among students, lack of interaction and disinterestedness in learning. In view of this, there is concern on the part of educators that low performance in mathematics may be attributed to the students' underdeveloped state of conceptual understanding as they continue relying more on rote memorisation. In our teaching practices, this is evident in students' struggle to effectively establish connections between previously acquired knowledge and newly introduced concepts, in the hindered interrelations of various topics and strands within the discipline as well as in the challenges they face when applying familiar concepts in new settings. In such instances, students tend to experience a heightened sense of anxiety leading them to circumvent questions perceived as intricate or challenging. This underscores the importance of nurturing in our students a belief in their capacity to forge mathematical connections that leverage prior knowledge, thereby helping them to succeed in learning new concepts. If we instil in students the ability to consistently inquire and investigate, then this will facilitate the identification of mathematical patterns, the recognition of having a web of connected concepts and the ability to formulate generalizations within the discipline (Calleja, 2017).

As outlined in the literature (Eli, Mohr-Schroeder, & Lee, 2013; Hidayati & Kurniasari, 2021, Sullivan, 2016 as cited in Klosterman, 2018), an enhanced mathematical connection ability correlates not only with improved retention of knowledge but also with a more nuanced understanding of when and how to effectively apply the acquired skills and content. Bingölbali and Coşkun (2016) urge us to comprehend and acknowledge that students' rigid compartmentalisation of mathematical concepts might be attributed to the instructional methods employed by teachers. Our tendency to limit explanations to specific contexts within a given unit, coupled with the practice of teaching distinct strands and units in isolation, could be significantly contributing to this compartmentalisation. Hence, it is crucial to reflect on teachers' role in making mathematical connections explicit and in critically examining the teaching methods we are currently employing. Indeed, this was the driving force behind designing the research lesson 'Investigating the Hexagon: a lesson of discovery and exploration'. The collaborative learning opportunity was a springboard to address and mitigate the factors that contribute to students' segmented understanding of mathematics as "students must be guided and encouraged in mathematics learning to develop the habit of thinking, looking and asking about connections" (Baiduri, Putri, & Alfani, 2020, p. 1527).

By taking an unconventional path and employing an innovative pedagogy, the primary objective of the lesson study team was to help students utilise their existing knowledge and through the formation of meaningful connections, enhance the organisation of knowledge they already possess, deepen their understanding of mathematical concepts and facilitate mastery of new ones. The team focused on the strand 'Shape, Space, and Measure' due to the infinite, intricate interrelationships between its concepts as well as across other strands. Recognising the potential monotony that could ensue from a prolonged and repetitive exploration of concepts and principles using conventional methods, we decided to deviate from our established transmission type classroom practices and explore alternative approaches that could work for our students, and which aligned with our goals.

Additionally, this lesson sought to engage students in:

- collaborative learning and peer tutoring
- an active and participatory learning process
- developing self-directed learning to minimise reliance on the teacher
- presenting, discussing, and merging their investigative ideas
- valuing the method, strategy and process over their final answer.

**The lesson study context**

The lesson study team comprised of four mathematics teachers, including the first author, and the head of department (HoD). The teachers who joined in this lesson study were all teaching senior students at this state secondary school. The collegiality amongst teachers and the HoD accentuated their role as "knowledgeable practitioners" (Stepanek et al., 2006, p.1) as it allowed the exchange of insights into closely related curriculum challenges, shared difficulties in student learning and the implementation of new syllabi. A one-to-one LSE in the chosen class was also consulted during the process to ascertain that the student under his support was provided with the necessary assistance. The engagement of stakeholders at all levels generated a wealth of diverse perspectives whilst the harmonious blend of teaching styles and personalities enriched the lesson study team's repertoire of pedagogical approaches, ensuring an effective professional collaboration. To further amplify this learning experience, a senior lecturer in mathematics education at the University of Malta, and a team member of Collaborative Lesson Study

Malta (CLeStuM – www.clestum.eu), was invited to observe the lesson and provide feedback during the post-lesson discussion.

In the local context, we always had a hierarchical approach that restrained students from being subjects of their learning experience, failing to give them a degree of autonomy and an experience of explorative wonder. As stressed by Freire (2005), who encouraged a shift from the banking model of education, we felt the need to start prioritising open tasks which have collaborative inquiry as their pivotal point. Instead of giving students a downflow of explicit knowledge, limiting them to a bubble of understanding, we sought to inspire students to think outside the box, explore the unknown, question things and challenge perspectives. Indeed, our choice of thoroughly designing a lesson for a Year 10 high-ability class stems from the practical challenges teachers face as they grapple with time constraints and overloaded syllabi. Year 10 stands as a pivotal stage where students are expected to move beyond the basics of Year 9 and construct a robust and deep understanding of concepts through association and application of various ideas and procedures. However, the expansive curriculum necessitates a swift pace, leaving little room for in-depth exploration.

Moreover, the selected class struggled mostly when confronted with questions that lacked an obvious solution strategy and when various units covered in class were presented simultaneously in unfamiliar contexts. While this class excelled in carrying out procedures and methods in isolation, they usually encountered challenges when applying their knowledge in a broader, more integrated context. This class also presented the lesson study team with the challenge that students were not accustomed to being observed, raising concerns about how their behaviour and engagement might be influenced during the lesson.

**The lesson study process**

Once the lesson study team was formed, a tentative timeline for preparing and delivering the research lesson was established. Despite teachers' regular duties and demanding schedules, the team found and agreed upon a common 40-minutes weekly time slot for lesson study meetings. Recognising that a rigorous lesson study requires an investment of substantial time for research, discussion, planning, resource preparation and analysis, the team was aware that time was limited. Therefore, an online chat platform was established to provide ample space for an ongoing conversation between meetings. In the meantime, the lesson study kicked off with the team exploring the teaching and learning challenges encountered in their classrooms. While some members were contemplating revolving this research lesson on imparting effective memorisation techniques, others leaned towards designing a lesson that could help students recall and build upon existing knowledge through a connection of topics.

Following deliberations on the syllabus, scheme of work, and available instructional materials, the team decided to formulate and develop a lesson plan centred around geometrical topics. Each team member assumed the responsibility of researching and developing a variety of geometrical problems or tasks with the explicit goal of addressing the identified learning concern. After a comparative evaluation of the compiled tasks, a problem-solving task stood out as particularly effective. Recognising the diverse abilities within this class, the team sought to differentiate the task in a way to create a personalised learning experience that caters to all learners.

However, the team expressed dissatisfaction with this problem-solving task, noting that it retained characteristics resembling an exam-style question. In a pursuit to provide students with a non-traditional lesson, the task was transformed into an open-ended investigation which necessitated further discussions from the teachers' end to fine-tune it. Anticipating potential confusion among students, the team discussed at length how to assist students in structuring their inquiry and conducting more thorough investigations. Various possible approaches and potential outcomes for this open investigation were considered and sets of prompts for each scenario were prepared accordingly. Despite anticipating various scenarios, the team remained uncertain about the specific outcomes that students would generate. Consequently, this posed a challenging aspect for setting the stage for the teacher delivering the lesson to orchestrate an engaging 'neriage' (closure to the lesson) where students were allowed to direct the lesson based on their respective discoveries. Throughout the planning, the team actively sought the opinions and advice of various knowledgeable others. Feedback received, particularly regarding the challenging nature of the lesson for the teacher delivering it, prompted a reassessment of the lesson plan.

The lesson study began in early November 2023 and the research lesson was conducted in mid-January 2024, with the entire team present as observers along with the invited university-based mathematics teacher educator. On the same day the lesson was taught, the team along with the invited knowledgeable other, gathered for a post-lesson debriefing session. The session began with the facilitator outlining the meeting's agenda, followed by the class teacher who shared her initial reflections on the teaching of the lesson. This self-reflection allowed the teacher to set the stage for a collaborative exchange of insights where the observers used their data collection sheets to provide thorough feedback for an ongoing refinement of practices. The invited expert's contributions were also instrumental in assisting the team to pinpoint and elucidate potential blind spots as well as in reflecting on viable strategies to implement. Eventually, a collective engagement ensued, with all observers sharing suggestions aimed at augmenting the lesson's effectiveness. This post-debriefing session offered the team newfound perspectives on both the strengths and weaknesses inherent in the lesson. It emerged that students were highly engaged in exploring various methods and went beyond merely stating the facts, demonstrating that they could think in a creative way and construct their own knowledge outside of conventional boundaries.

**Key instructional decisions within the research lesson**

The research lesson embarked students on a journey of discovery and exploration as they delved into the intricacies of a regular hexagon featuring six equal shaded sectors at its vertices (see Figure 1). Designed as an open investigation, students were given the opportunity to formulate their own questions, explore the geometric properties of the hexagon, and creatively devise ways or solutions to address their curiosities. Acknowledging that this might potentially mark students' first open-ended mathematics investigation, we opted to use the 5E Inquiry-based Instructional model (Bybee & Landes, 1990) as a planning tool as well as a framework to guide students during the learning process. Anticipating an initial sense of discomfort among students, the teacher played a crucial role in providing support and guiding the students in the autonomous construction of knowledge by using effective questioning.

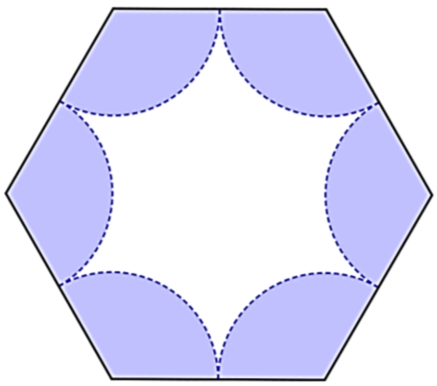


*Figure 1: The geometrical shape students investigated during the lesson*

Initially, students were allowed to engage with the investigation on their own, guided by a mind map (see Figure 2) to stimulate the retrieval of their prior knowledge and promote creative thinking. This initial phase provided students with the autonomy to explore the hexagon cutout in a manner aligned with their individual learning styles and preferences. Such approach gradually immersed students in the investigative process, allowing them to engage with discovery learning by brainstorming their ideas on their own.

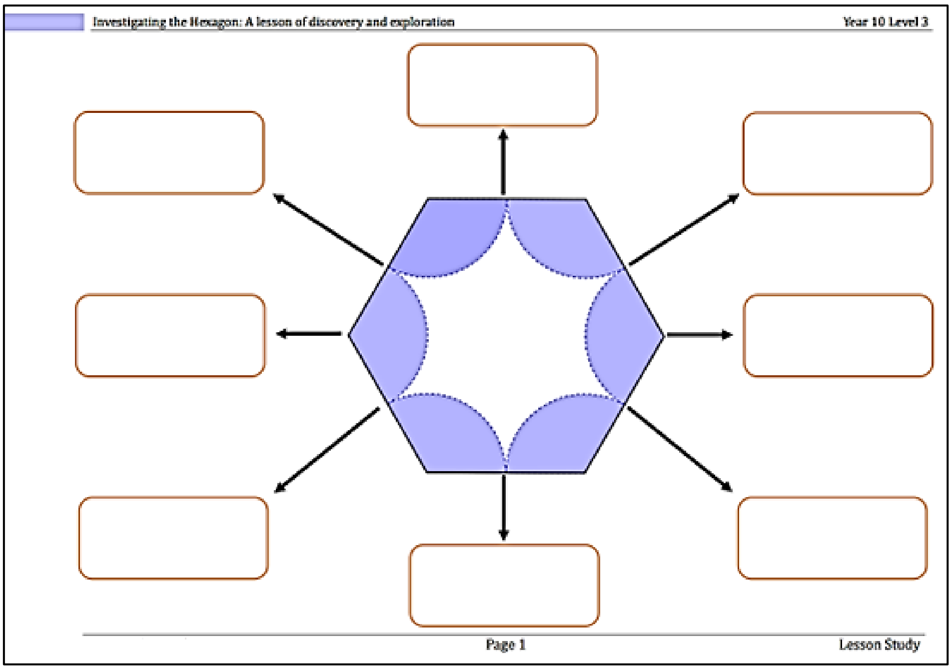



*Figure 2: The mind map used for the brainstorming activity*

Subsequently, students collaborated in small groups to explore and explain in greater depth a specific, distinct feature of the provided figure. Students were encouraged to develop a plan (see Figure 3) that outlines the approach they took to investigate their chosen feature and to evaluate its significance. The plan was expected to incorporate relevant mathematical facts, concepts, and principles that would serve as the foundation for their investigation. This analytical exercise provided a platform for students to encounter various mathematical concepts, including topics like polygons, angles, trigonometric ratios, Pythagoras' theorem, area, circles, and/or tessellations and to broaden their understanding of geometry as they link these various topics. Moreover, the collaborative work served not only as a gateway for students to discover new concepts, such as calculating the area of sectors, but also for students to consider alternative approaches and explore different methods. Given that this lesson was designed and planned with the intention of alleviating students from the constraints of closed questions, we predicted that such approach will be able to foster a variety of creative presentations.

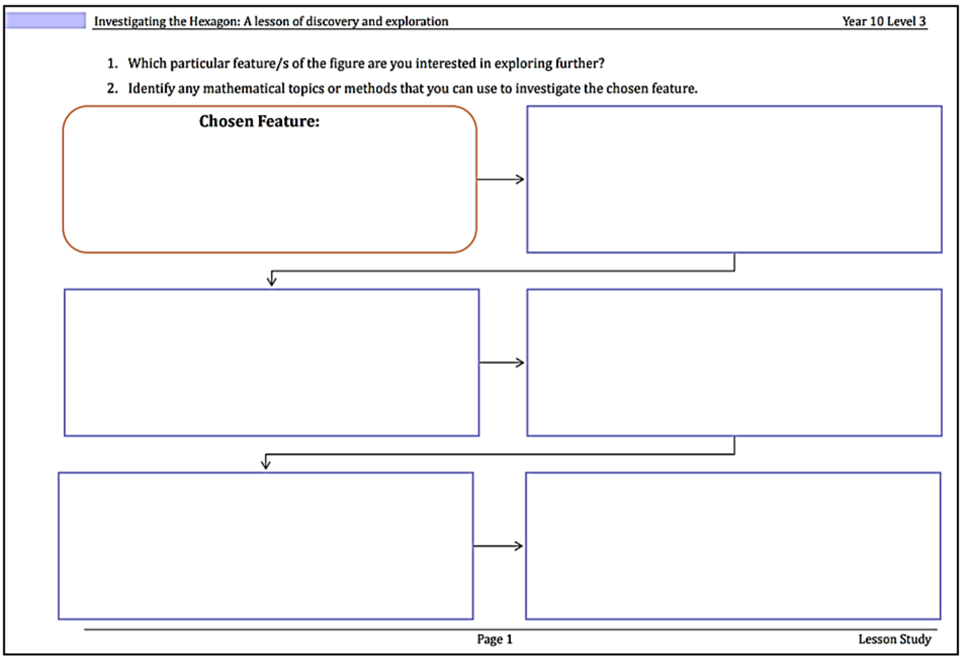



*Figure 3: The concept plan worksheet used during group work*

Later, the rapporteurs from each group presented their investigations to the entire class, demonstrating the evolution of their thinking from the initial stages (Takahashi, 2008) and exposing the class to a broader spectrum of mathematical concepts and methods for a rich and multifaceted learning experience. Throughout the entire lesson, students were encouraged to adopt a mindset of inquiry, akin to that of a detective, fostering a sense of curiosity and a commitment to uncover the underlying principles governing the mathematical concepts at hand. A detailed lesson plan can be found in the Appendices.

**Highlights documenting the lesson study process**

In this article, a series of photos that illustrate the various phases of the designed research lesson are presented. These snapshots showcase everything from the lesson's introduction and individual brainstorming activity to the collaborative group work, students' presentations, and the whole class discussion that took place during the concluding phase of the lesson. Alongside these, exemplars of students' mind maps used for brainstorming as well as the concept plans devised by the groups are displayed, offering a broad outlook of both the learning process and the outcomes achieved. The feedback gathered through the observers' data collection sheets and students' reflection sheets is also shared. Ultimately, a photo of the lesson study team during a meeting, engaged in discussing and making the final adjustments to the lesson plan is included.

Photographs of the lesson:

*Figure 4: The class teacher introducing the lesson*


*Figure 5: The class teacher introducing the investigation*

*Figure 6: Individual brainstorming activity*

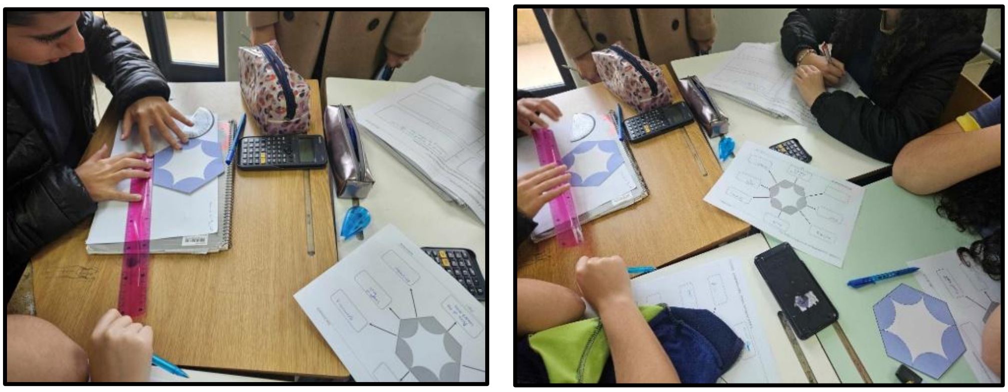


*Figure 7: During the collaborative work, a group of students discussing their concept plan*

*Figure 8: The lesson study team and invited expert observing the lesson*

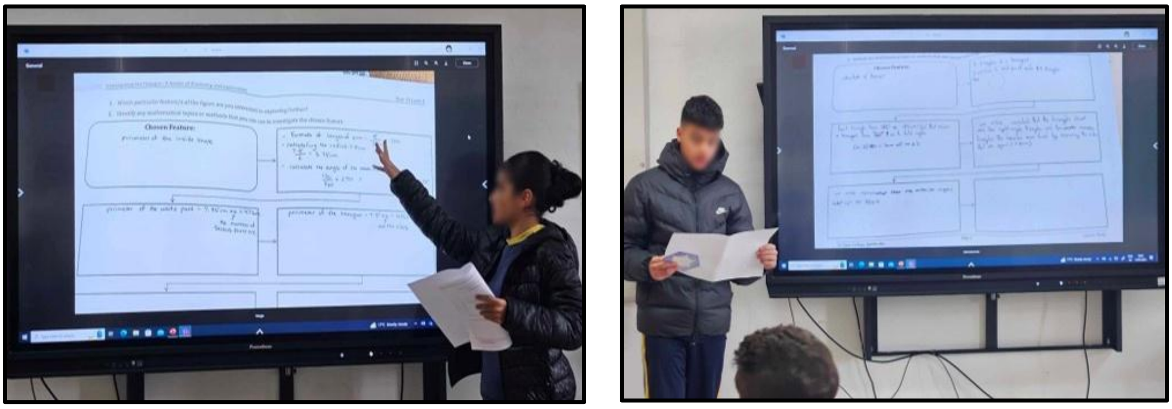


*Figure 9: Student rapporteurs presenting their group work*

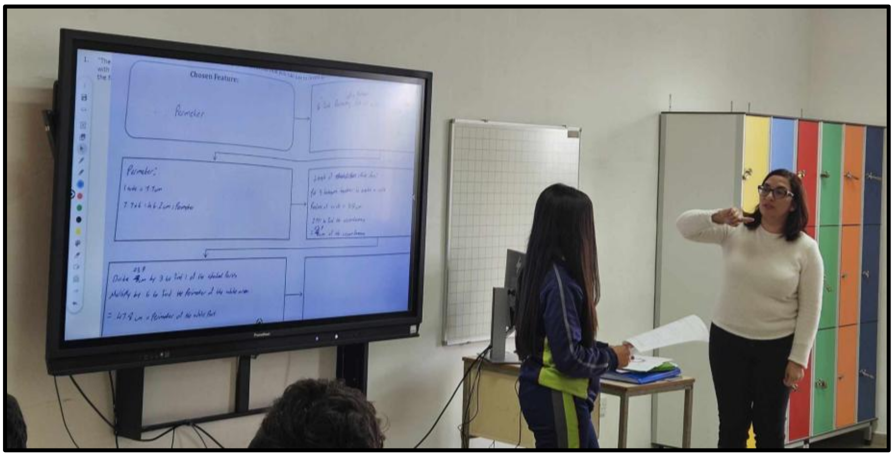
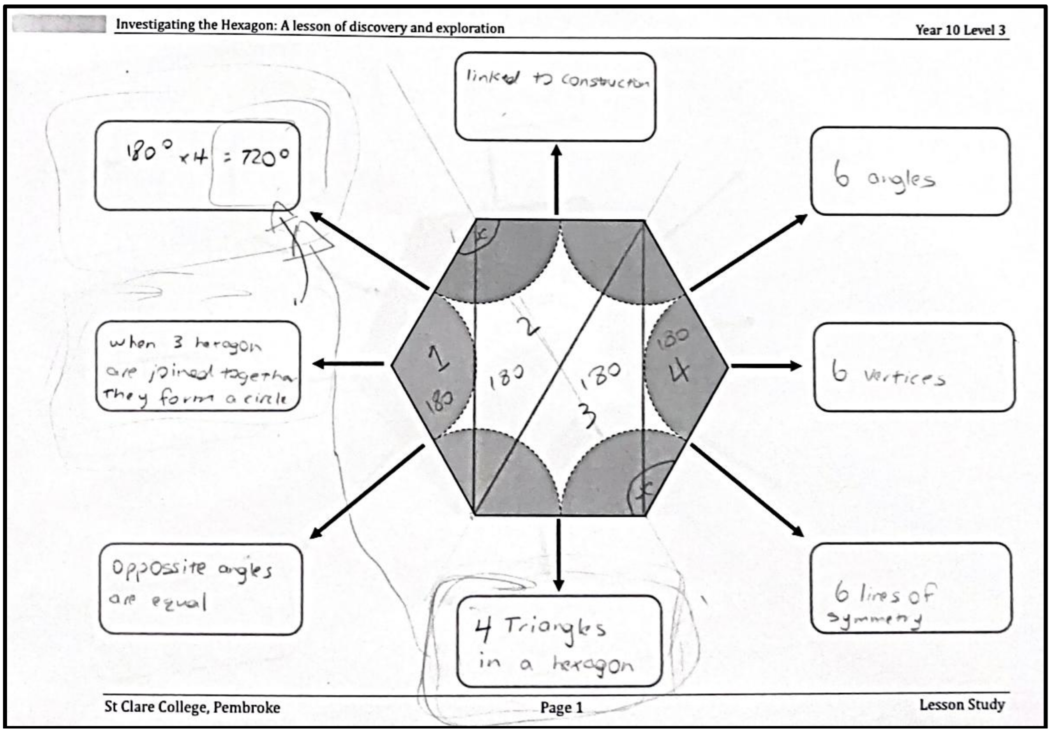



*Figure 10: A whole class discussion led by the teacher*

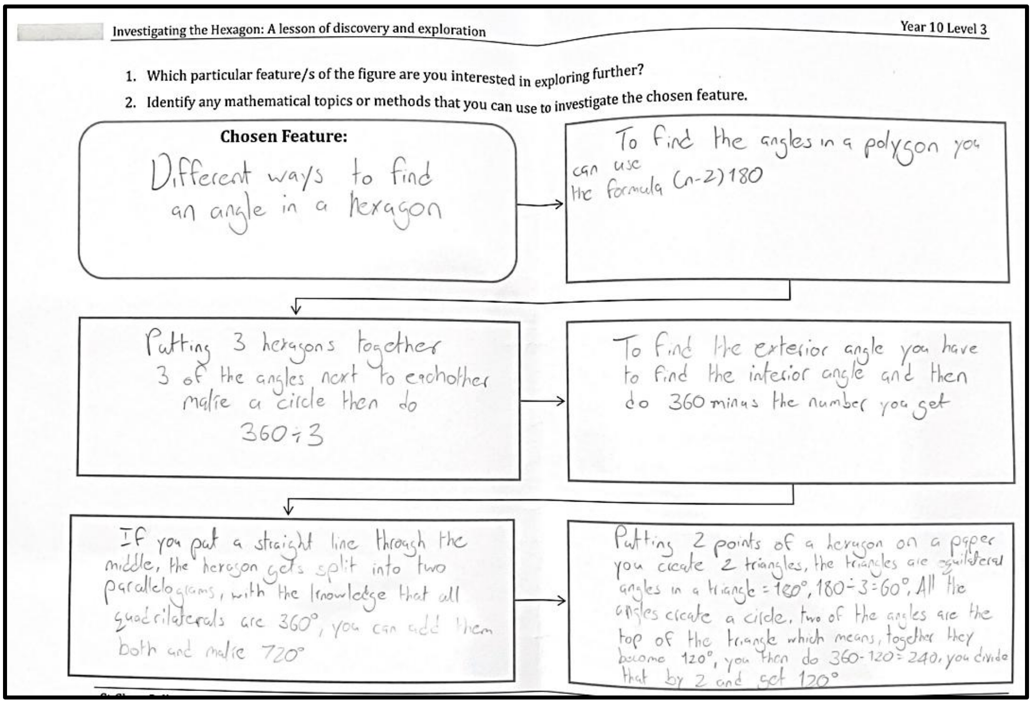
*Figure 11: Concluding the lesson*

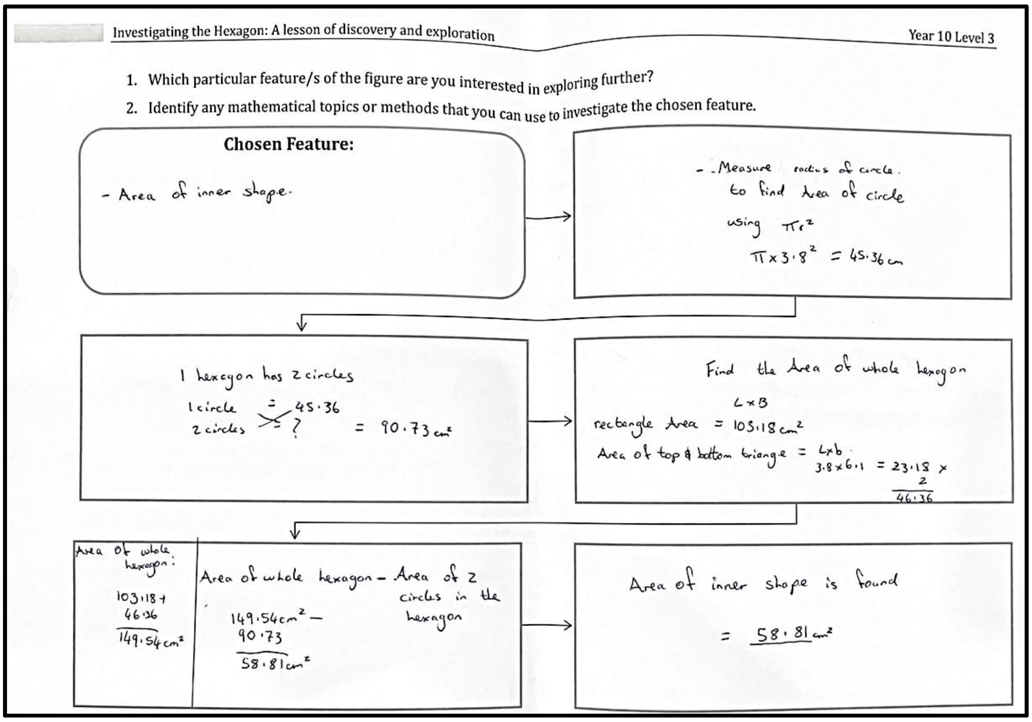
*Figure 12: The lesson study team during a meeting*

An exemplar of students' mind maps:



Two exemplars of students' concept plans:



Students’ reflection sheet:

# Reflection Sheet

| | |
|---|---|
| *What struck you most from this lesson?* | The groupwork and brainstorming |
| *What did you learn from this lesson?* | I learned how ~~the~~ to find the Area, perimeter of a hexagon and I learned how to work as a group |
| *Were there any specific moments during the lesson where you felt particularly engaged?* | when it came to doing groupwork and listening to each others ideas |
| *What was the most challenging aspect of this lesson?* | To explain and present the work |
| *Do you still have any questions or concerns about this lesson?* | NO, ~~because~~ |
| *Is there anything you'd like to see done differently in future lessons?* | More teamwork |

Example of data collected from one of the lesson observers:



## Observer Evaluation Sheet

**Looking for evidence:**

*What do students do? What questions do they ask? How do students respond to the questions posed? What do they discuss? What do students write? How do they react? What do students struggle with?*

| Activity 1: Introducing the 5Es and Presenting the Investigation | | | |
|---|---|---|---|
| **Students...** | **Yes** | **No** | **How is it evident?** |
| *Listen attentively to what the teacher is explaining.* | ✓ | | |
| *Understand the skills needed for the investigation.* | ? | | no questions are asked but does it mean that they understood? |
| *Clearly understand the investigation and what is expected from them.* | ✓ | | some of them sought help from their friends; asked questions to their friends' and not the teacher. |

| Activity 2: The mind map – Individual Work | | | |
|---|---|---|---|
| **Students...** | **Yes** | **No** | **How is it evident?** |
| *Take the initiative to use mathematical tools to acquire information from the diagram.* | ✓ | | use of protractors and rulers to measure the length of sides and any unknown angles. |
| *Mention facts about the hexagon given.* | ✓ | | mentioned basic facts (e.g no. of sides) |
| *Identify concepts related to the hexagon.* | ✓ | | mentioned area, polygons and constructions. |

| Activity 3: The mind map – Whole Class Discussion | | | |
|---|---|---|---|
| **Students...** | **Yes** | **No** | **How is it evident?** |
| *Offer ideas that link with the hexagonal cut-out given.* | ✓ | | all the students in the group contributed and shared ideas. |
| *Successfully record their observations on the mind map.* | ✓ | | Students managed to record several observations |
| *Recall prior knowledge related to the hexagon given.* | ✓ | | mentioned concepts covered in year 9 and discussed the methods and formulae learnt. |

*Observer 1:*



| Activity 4: Group Work – Devise a Plan | | | |
|---|---|---|---|
| **Students...** | **Yes** | **No** | **How is it evident?** |
| *Understand the second part of the investigation and what is expected.* | ✓ | | the group started discussing straight away which aspect to focus on. |
| *Establish connections between various concepts/principles/topics.* | ✓ | | they connected perimeter, circles, sectors and polygons. |
| *Manage to find missing information.* | ✓ | | measured the unknown angles and the radius of the sectors. |
| *Contribute ideas to the group and are fully focused.* | ✓ | | managed to find the arc length despite forgetting the formula. |
| *Provide well-developed mathematical arguments.* | ✓ | | at first they were considering the sectors as semi-circles. |

| Activity 5: Presentation and Whole Class Discussion | | | |
|---|---|---|---|
| **Students...** | **Yes** | **No** | **How is it evident?** |
| *Present coherently and clearly the plan to the entire class.* | ✓ | | the presenter was a little bit shy but then managed to explain well the group's work. |
| *Communicate the connections they established, the methods used and the underlying reasons.* | ✓ | | the presenter explained clearly the process of their investigation, i.e. step by step how they found the perimeter of the white part. |
| *Answer questions that might arise from the teacher or other students.* | | | no questions were asked to this group. |
| *Make connections between the investigations presented.* | ✓ | | teacher connected the investigations. students were more concerned about their respective investigations. |

| Activity 6: Teacher Summary | | | |
|---|---|---|---|
| **Students...** | **Yes** | **No** | **How is it evident?** |
| *Understand the significance of the process rather than solely focusing on achieving a definitive solution.* | ✓ | | all the students presented a concept plan with a detailed process. teacher insisted on the process and emphasized the connection of topics. |
| *Appreciate that the investigation was open-ended, allowing them to pursue any area of interest they desired.* | ✓ | | |

**Main takeaways from teaching the research lesson**

The lesson study members expressed an overall satisfaction with the lesson, noting that its main aims and objectives were successfully achieved. Nonetheless, several areas for improvement were identified and discussed. Although the lesson's structure deviated from the typical format, it still succeeds in effectively capturing and maintaining the students' attention as a variety of activities were incorporated. Despite typically having a short attention span, these students exhibited genuine interest and managed to generate intriguing investigations, effectively utilising their prior knowledge, and connecting various topics and concepts. However, the team observed that the lesson's introduction may have overwhelmed some students due to the amount of theoretical information presented, contributing to initial hesitation and uncertainty in carrying out the investigation. Moreover, some teachers remarked that the investigation itself might have been too demanding and challenging, especially considering that it was their first attempt. Consequently, to address this, it was suggested to first introduce students to less demanding investigations using inquiry-based learning strategies before gradually progressing to more challenging and open investigative tasks. It was also recommended to alleviate students' apprehension by consistently reminding them that the focus should be on the reasoning process rather than on achieving a specific outcome, while simultaneously acknowledging their ongoing efforts.

The observers, taking on the role of co-teachers during the collaborative work, providing the necessary support and guidance, played a key role in fostering a conducive environment for students. Given that the groups of students were concurrently involved in various investigations, the assistance from the observers proved also invaluable for the class teacher to effectively manage all groups. Moreover, the thorough planning, anticipation of possible investigations together with the preparation of questioning prompts enabled the teacher and observers to track the students' work effectively. The open-ended nature of the investigative task also necessitated an extensive, detailed lesson plan, as multiple scenarios needed to be anticipated, based on how each respective group might approach the task. Corresponding questioning prompts were also prepared to accommodate different approaches. Indeed, when this lesson plan was initially presented to the knowledgeable other for feedback, it appeared nearly impractical to deliver as written. However, after observing the lesson in action, the invited expert gained a clearer understanding of its design and structure. The lesson study team also acknowledged that given its nature, this research lesson could easily unfold in a completely different way with other classes. While the team emphasised the importance of anticipating potential outcomes and being well-prepared with questioning prompts, they also expressed their reservations about adhering to a scripted lesson plan, particularly in the context of a lesson that revolves around an open investigation where the teacher's flexibility is crucial.

Finally, time constraints proved to be a significant challenge for the lesson study team. The lesson, originally scheduled for 40 minutes, was extended by an additional 20 minutes when it was discovered that students had a free period following this lesson. Although the team swiftly adapted to utilise this extra time, not all observers could stay for the entire extended period due to other commitments. Reflecting on this, we discussed how allocating additional time for collaborative work during the planning phase could have been beneficial, given the students' eagerness to continue their investigations. This adjustment could have potentially led to more interesting, creative, and detailed presentations. Additionally, optimising time distribution across each stage of the lesson would ensure that the teacher has enough time to provide clear instructions, involve more students in the class discussions and effectively highlight the connections between the investigations presented.

**Closing Remarks**

As lesson study anchors itself firmly in the practical realities of classroom instruction and the intricate dynamics of student learning (Lewis, 2009), it is highly valued by many teachers who get involved in it. This grounded approach leads to a more comprehensive understanding of pedagogical principle, the challenges and opportunities that arise in the classroom daily. In fact, the lesson study team members acknowledged how such collaborative approach enabled them to create a lesson that exceeded their individual capacities. They felt that the collaborative nature of the process encouraged the exchange of ideas, stepping out of their comfort zones, consideration of different perspectives and engagement in reflective practices. These aspects led to the realisation that there were unexplored possibilities in how such a lesson can be planned and delivered and although the lesson design went through various changes, the collaborative discussions were worthwhile to eventually plan a student-centred lesson.

This lesson study experience taught the team members that, at times, it is teachers who intentionally or unintentionally impose limitations on students' learning by hesitating to challenge them and by not fully trusting their capacity to drive their own learning. Indeed, this team remained quite impressed by the students' presentations as these not only proved to be exceptionally interesting, but some investigations turned out to be completely unexpected. As put forward by Kennedy (2005, p. 246), when educators are strongly involved in researching and delving into what works for their students, research will not be any more a "product of someone else' endeavours", but it is intertwined in the daily activities narrowing the gap between theory and practice. The researched goal is tested and analysed by teachers in the field making them the actual source of new, authentic professional knowledge which can be shared with others as in this case. This expands the capacity of teachers to lead and pursue their own professional growth rendering them more accountable in guiding students down the path of success.

The transition of the first author, from being a participant to assuming a leadership role, particularly among colleagues with whom she had previously collaborated in lesson studies, prompted her to reflect on the evolving role and leadership capabilities. With hindsight, this experience turned out to be a valuable opportunity for self-reflection, encouraging her to evaluate her leadership skills and view the collaborative process from a new perspective. Also, the lesson study cycle proved to be instrumental for experimenting with new pedagogies, engaging in active learning and in strengthening the team's professional learning community. While working conjointly can be challenging due to teachers' differing concerns and/or beliefs, this powerful and enriching approach to PD generated a more relaxed atmosphere for teachers to discuss teaching and learning challenges, to seek advice, to reveal struggles and to explore innovative teaching methods.

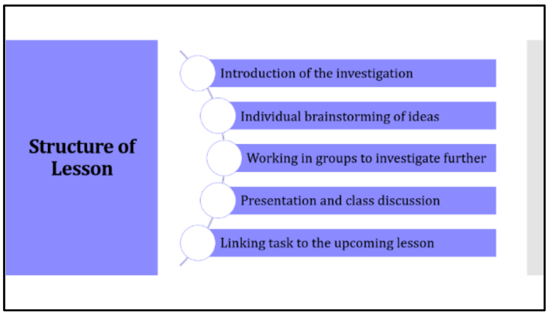
Looking ahead, while the lesson study team envisions a continuing participation in lesson studies and other collaborative projects, they also hope to expand this practice by involving teachers from other departments. Furthermore, the team is excited about the possibility of scaling up this initiative to a whole-school PD approach.

**Acknowledgment**

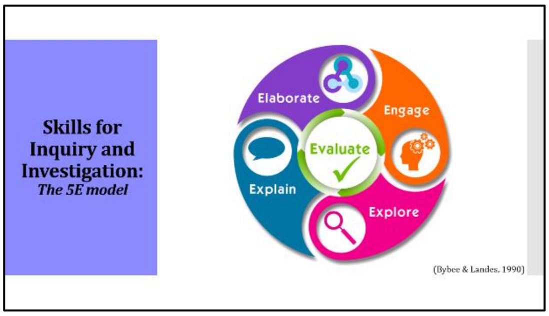
Most of all, I would like to express my sincere gratitude to the mathematics teachers and the LSE who joined this lesson study. Their generous help and vast pool of shared ideas were truly instrumental in designing this research lesson in great depth. Their wholehearted dedication and deep commitment were above and beyond expectations, rendering this collaborative learning experience a rewarding one. We would also like to extend thanks to the HoD and the invited knowledgeable other for their constructive feedback, insightful suggestions and passion for enhancing teaching and learning.

**Appendices**

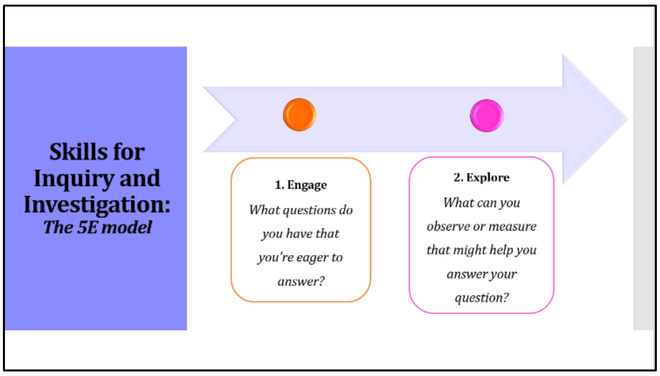
Lesson Plan for Investigating the Hexagon: A lesson of discovery and exploration

| **Introduction** | |
|---|---|
| ***Step 1: Welcoming students and explaining the observers' roles*** | |
| **Time:** | ***1 minute*** |
| ▪ Upon entering the classroom, the teacher accentuates the importance that students join the pre-determined groups of four.<br>▪ The lesson begins with the teacher giving a warm welcome to the students, introducing the observers and explaining their roles in conducting a lesson study. | |
| ***Step 2: Introducing and explaining the 5E's model*** | |
| **Time:** | ***3 minutes*** |
| ▪ Utilising a PowerPoint Presentation, the teacher indicates that this lesson is **a journey of discovery and exploration, focusing on the investigation of a hexagon**. To captivate students' attention, the teacher will display a slide with images of a nut, honeycomb, and the iconic James Webb Space telescope.  <br>▪ Following this, the teacher **outlines the main steps** of the lesson.   | |

- Recognising that this might be the first time for these students to carry out a mathematical investigation, the teacher introduces the essential skills required for inquiry and investigation by employing the well-established 5Es framework. This structured approach encompasses the five distinct phases: **engage, explore, explain, elaborate, and evaluate**.



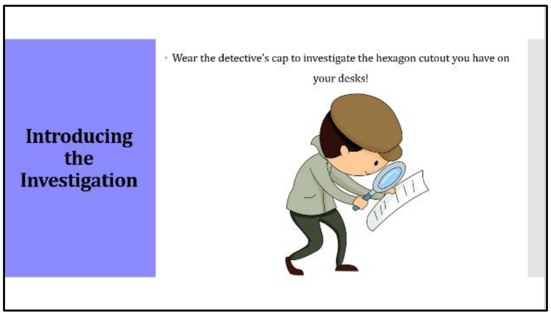

- The teacher remarks:
  ***"Here, we will engage with an investigation and try to explore it. To do this, we are going to ask ourselves:***
  1. ***What questions do I have that I would like to answer?***
  2. ***What can I observe and measure that might help me answer my questions and curiosities?"***




**Step 3: Presenting the task and the role of students**

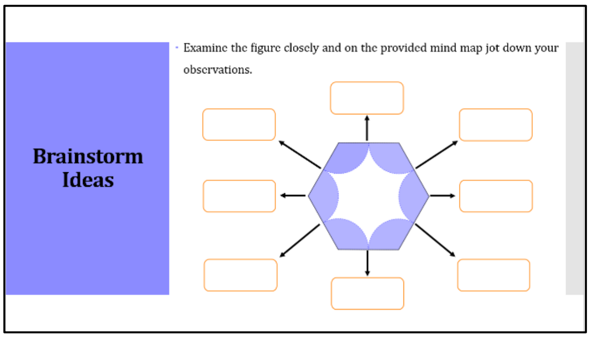
| **Time:** | ***6 minutes*** |
|---|---|

- The teacher introduces the open investigation with the metaphor '**wearing the cap of a detective'.** This analogy serves as a powerful reminder that this task is not about seeking a single, definite answer but its objective is to actively explore, question, collect information, establish connections, and discover new knowledge.

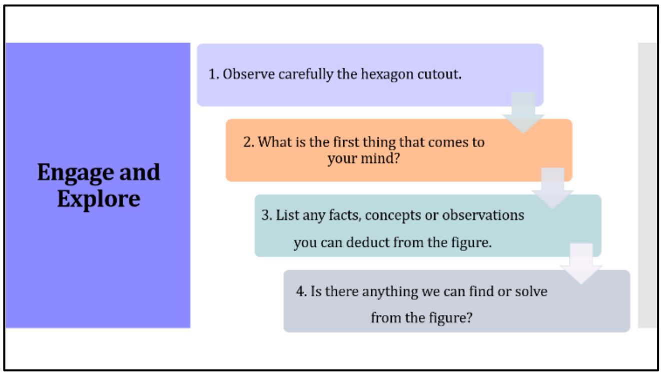



- Students are urged to individually **conduct a close examination of the hexagonal-shaped cutouts** provided and encouraged to think outside the box as they delve into the various aspects and features of the hexagon. The teacher presents the task and explains:

*"Remember, as investigators, we don't have limits. You can jot down on the provided mind map anything that sparks your curiosity, even if it seems unrelated at first. Be creative and don't be afraid to challenge yourself".*

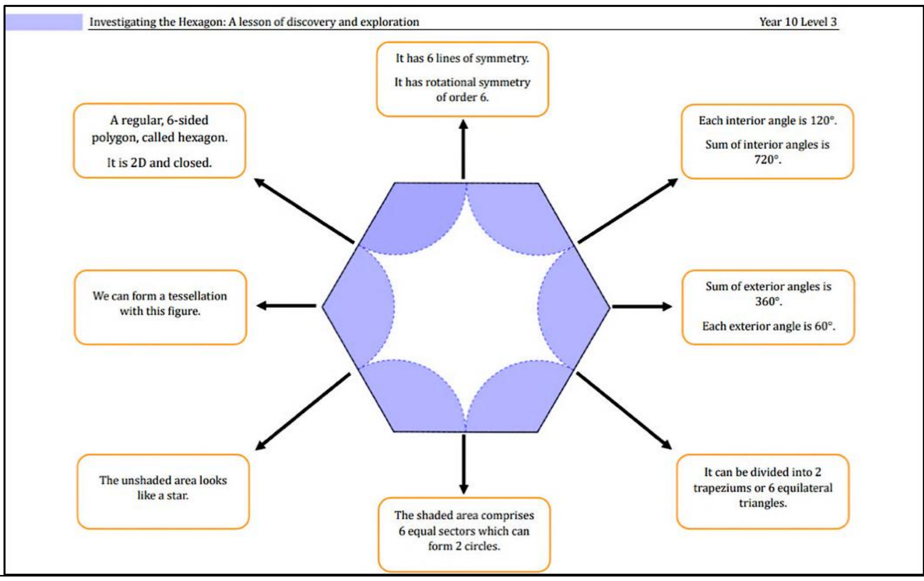



- The teacher will display a slide containing four guiding questions to assist students in navigating the investigation and formulating meaningful inquiries.



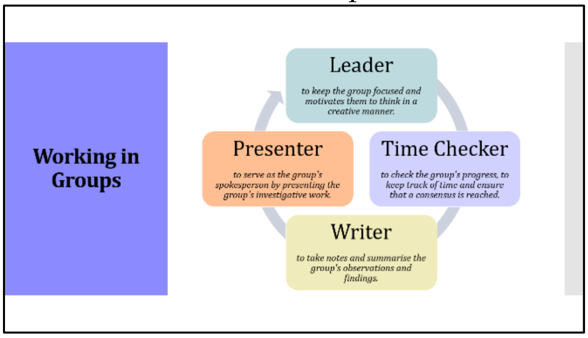

- After providing time for the students to engage and explore the task individually, the teacher invites some students to share any ideas that surfaced during this brainstorming session.

- Here are some anticipated observations that students might record on their mind maps:

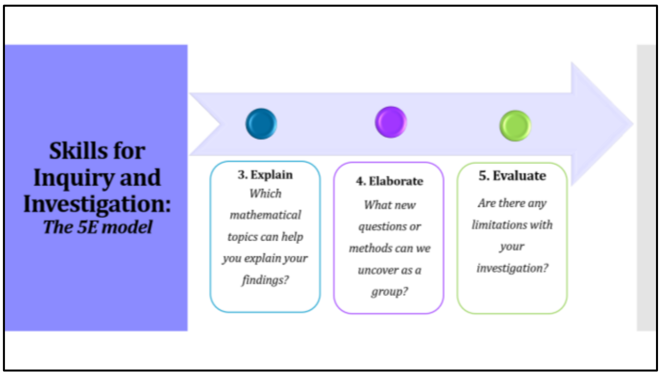

Investigating the Hexagon: A lesson of discovery and exploration
Year 10 Level 3
It has 6 lines of symmetry.
It has rotational symmetry of order 6.
A regular, 6-sided polygon, called hexagon.
It is 2D and closed.
Each interior angle is 120°.
Sum of interior angles is 720°.
We can form a tessellation with this figure.
Sum of exterior angles is 360°.
Each exterior angle is 60°.
The unshaded area looks like a star.
It can be divided into 2 trapeziums or 6 equilateral triangles.
The shaded area comprises 6 equal sectors which can form 2 circles.

| Development *(Part 1)* | |
|---|---|
| **Step 1: Explaining students' roles and using the 5E's model** | |
| **Time:** | *2 minutes* |

- Following a brief discussion on the individual mind maps created, the teacher informs the students that for the upcoming part of the lesson, they will be required to work in the pre-assigned groups of four.

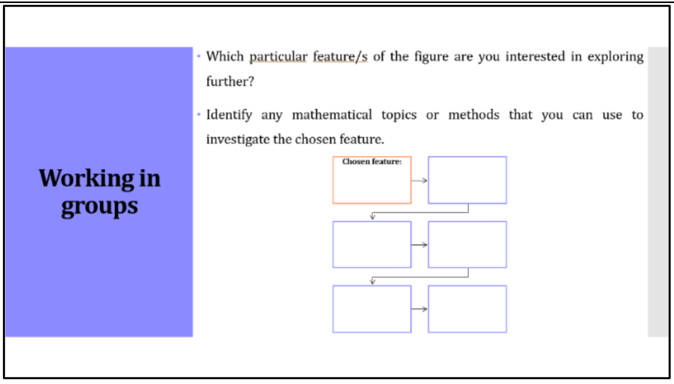



- Via the PowerPoint Presentation, the teacher reviews the 5E model with the students and explains:
  ***"In groups, you are going to compare your mind maps, explain and elaborate more on them. Whilst working in groups you are going to keep in mind these questions:***
  ***Which mathematical topics can help us explain what we are investigating?***
  ***What new questions or methods can we uncover as a group?"***

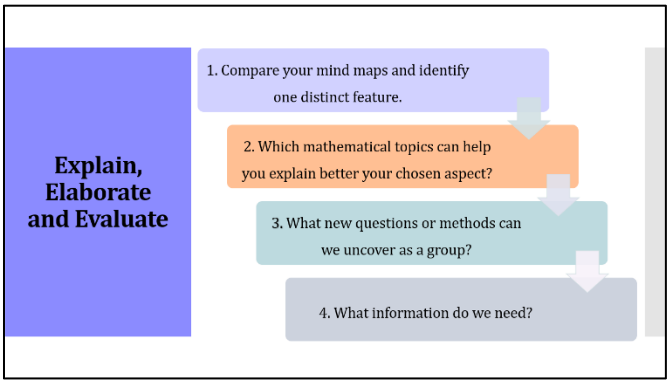



| **Step 2: Presenting the second part of the task** | |
|---|---|
| **Time:** | *2 minutes* |

- The teacher introduces the second part of this investigation and explains what is expected from each group at this stage.
- Students are asked to exchange and compare their individual mind maps to identify a distinctive feature worth delving into further. The focus is on illuminating the methods, concepts, topics, or principles necessary to add more layers of depth and exploration.




- The teacher proceeds:
  ***"Remember that what's important is the process of investigation not the final answer. In the separate boxes, write down all the steps you will need to follow.***
- To support students during the group work, the teacher says:
  ***"On this slide you have some questions that might help you devise the plan. Please, raise your hands if there are any queries".***




- The teacher will present a timer to help students adhere to the allotted time.

| **Step 3: Collaborative work** | |
|---|---|
| **Time:** | ***12 minutes*** |

- While students are engaged in collaborative work, the teacher attentively observes the initial progress and attempts to discern the distinctive feature each group is exploring.
- Sets of open-ended questions were prepared for each scenario. These questions serve as prompts to help the teacher and observers to guide the groups towards the next step of their investigation.

**Area of unshaded region:**

- What other geometric shapes can you find inside the hexagon?
- How can you split up the hexagon into basic shapes?
- Do you have enough information to find the area of these shapes? If not, what information is missing?
- Can we find the missing information using the knowledge you already know about?
- If we are unable to find the missing information, can we adopt a different approach to obtain the information?

| Development *(Part 2)* | |
|---|---|
| **Step 1: Students' presentation of work** | |
| **Time:** | ***12 minutes*** |

- The teacher will allocate approximately 2 minutes per group for their presentation.

- During the presentations, students will create a whole class discussion and each group's spokesperson is expected to take the lead in articulating the group's work clearly, providing explanations of the methods and concepts used.

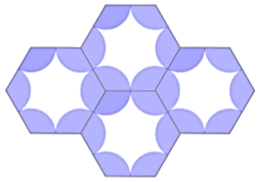
- Following the presentation of each group, the teacher asks the rest of the class: ***"Is there anyone who would like to add anything related to this investigation, or perhaps someone explored something similar?"***

- The following are some intriguing features of the hexagonal-shaped cutout that we are anticipating that the groups will explore and present. Nevertheless, students have the flexibility to investigate additional or different features than those outline here. The direction of the lesson is in the students' hand, entirely determined by their efforts and choices.

**Scenario 1:** *In case, a group opts to explore 'Tessellations':*

**Chosen Feature: Tessellations**

Shapes are arranged to fit closely together without any gaps or overlapping.

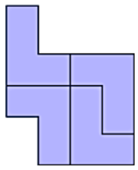


Hexagons tessellate as their interior angles of 120° is a factor of 360°.

Three hexagons can fit together. In this case, at each vertex a shaded circle is formed.

Hexagons will tessellate without any restrictions if they are regular, i.e. have equal sides and angles.

Irregular hexagons can also form tessellations if the arrangement of sides is modified, for instance as shown in the diagram.



Other polygons that tessellate are triangle, squares and some irregular polygons.

These tessellate as the size of one interior angle is a factor of 360°.

Even if regular, the pentagons do no tessellate as one interior angle of a regular pentagon is 108°.

360° is not a multiple of 108°, thus a tessellation cannot be created. There will be overlaps or gaps.

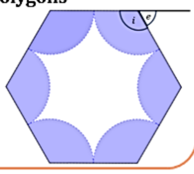
However, we can create semi-regular tessellations.

**Scenario 2:** *In case, a group opts to explore 'Polygons':*

**Chosen Feature: Polygons**

It is a regular hexagon with 7.5cm sides measured using a ruler.

The interior and exterior angles are equal.




To find the size of one interior angle:

$$sum\ of\ interior\ angles = 180(n-2)$$
$$sum\ of\ interior\ angles = 180(6-2)$$
$$sum\ of\ interior\ angles = 720°$$
$$one\ interior\ angle = \frac{720°}{6} = 120°$$

To find the sum of interior angles of the hexagon, we can split the shape into:

**2 trapeziums:** $sum\ of\ interior\ angles\ of\ quadrialterals = 360°$

$$sum\ of\ interior\ angles\ of\ hexagon = 360° \times 2 = 720°$$

**6 equilateral triangles:** $sum\ of\ interior\ angles\ of\ triangles = 180°$

$$sum\ of\ interior\ angles\ of\ hexagon = (180° \times 6) - 360° = 720°$$

To find the size of one exterior angle:

$$sum\ of\ exterior\ angles = 360°$$
$$one\ exterior\ angle = \frac{360°}{6} = 60°$$

Or:

$one\ exterior\ angle = 180° - 120° = 60°$ (angles on a straight line)

The angles can also be measured using a protractor.

By finding the interior and exterior angles, we can also calculate the shaded area.

This can help us as well in using trigonometric ratios to calculate the length of any unknown sides.

Knowing the interior and exterior angles, we can conclude whether a polygon tessellates or not.

Regular hexagons tessellate as their interior angles of 120° is a factor of 360°.

Three hexagons can fit together. In this case, at each vertex a shaded circle is formed.

## **Scenario 3:** *In case, a group opts to explore 'Perimeter':*

**Chosen Feature: Perimeter of the White Part**

To find the perimeter of the white part, we need to first find the length of each arc.

Arc is part of the circumference.

To find the circumference of a circle, we can use the formula:

$$C = 2\pi r$$

To find the arc length, we can use the formula:

$$Arc\ Length = \frac{\theta}{360}° \times 2\pi r$$

First, we need to find the radius and the angle inside the sector, i.e. the angle at the vertex.

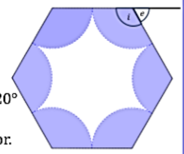
Since the hexagon is regular, all sides and angles are equal.

Radius = 4cm *(measured)*

Exterior angle of a hexagon = $\frac{360°}{6} = 60°$

Interior angle of hexagon = $180° - 60° = 120°$

The angle can be measured using a protractor.




$$Arc\ Length = \frac{120°}{360°} \times 2\pi \times 3.75$$
$$Arc\ length = 7.853\ ...cm$$

Perimeter = 6 equal arcs

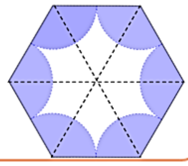
Perimeter = 7.853 ... × 6

Perimeter = 47.12*cm* *(to 2dp)*

## Scenario 4: *In case, a group opts to explore 'Area':*

**Chosen Feature: Area of Hexagon**

Split the hexagon into six equilateral triangles.



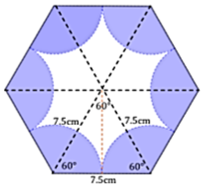
First, we need to find the area of one triangle, using the formula:

$$Area\ of\ triangle = \frac{b \times h}{2}$$

$base = 7.5cm\ (measured)$

To find the height of the triangle, first we must split the equilateral triangle in half to form two right-angled triangles.

Each angle is 60° since the triangles are equilateral.




To find the height of the triangle, we can use Pythagoras' theorem:

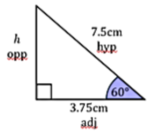
$$hyp^2 = side^2 + side^2$$

$$7.5^2 = h^2 + 3.75^2$$

$$\sqrt{7.5^2 - 3.75^2} = h$$

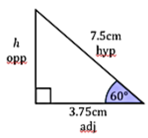
$$\sqrt{42.1875} = h$$

$$h = 6.495190528cm$$




To find the height of the triangle, we can use trigonometric ratios:

| | |
|---|---|
| $tan\theta = \frac{opp}{adj}$ | $sin\theta = \frac{opp}{hyp}$ |
| $tan60° = \frac{h}{3.75}$ | $sin60° = \frac{h}{7.5}$ |
| $3.75tan60° = h$ | $7.5sin60° = h$ |
| $6.495\ldots cm = h$ | $h = 6.495\ldots cm$ |




$$Area\ of\ one\ triangle = \frac{7.5 \times 6.495\ldots}{2} = 24.356\ldots cm^2$$

$$Area\ of\ hexagon = Area\ of\ 6\ traingles$$

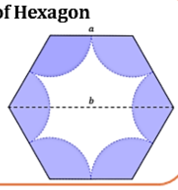
$$Area\ of\ hexagon = 24.356\ldots \times 6$$

$$Area\ of\ hexagon = 146.14cm^2\ (to\ 2dp)$$

## OR:

**Chosen Feature: Area of Hexagon**

Split the hexagon into two equal trapeziums.




To find the area of a trapezium, we can use the formula:

$$Area\ of\ trapezium = \frac{1}{2}(a + b)h$$

We need to find the length of the parallel sides and the perpendicular height of the trapezium.

Length of parallel sides:

$a = 7.5cm$

$b = 7.5 \times 2 = 15cm$

These can be measured using a ruler.

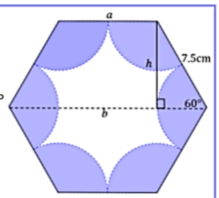
Hypotenuse = 7.5cm *(measured)*

Angle = 60° *(measured)*

Exterior angle of a hexagon = $\frac{360°}{6} = 60°$

Interior angle = $180° - 60° = 120°$

Angle = $\frac{120°}{2} = 60°$

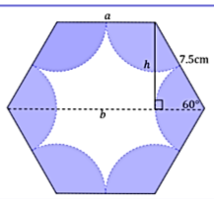



$$sin\theta = \frac{opp}{hyp}$$

$$sin60° = \frac{h}{7.5}$$

$$h = 7.5sin60°$$

$$h = 6.495\ldots cm$$




$$Area\ of\ trapezium = \frac{1}{2}(7.5 + 15) \times 6.495\ldots$$

$$Area\ of\ trapezium = 73.070\ldots cm^2$$

$$Area\ of\ hexagon = 73.070\ldots \times 2$$

$$Area\ of\ hexagon = 146.14cm^2\ (to\ 2dp)$$

## Scenario 5: *In case, a group opts to explore 'Shaded Area':*

**Chosen Feature: Shaded Area**

Shaded area = area of 6 equal sectors.

A sector is a fraction of a circle.

→

Area of sector = A fraction of the area of a circle

To find the area of a circle, we can use the formula:

$Area\ of\ circle = \pi r^2$

First, we need to find the length of the radius.

↓

Length of radius = 3.75cm *(measured)*

Area of circle = $\pi \times 3.75^2$

Area of circle = $44.178\ ...\ cm^2$

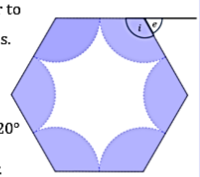
→

We need to find the angle inside the sector to find what fraction of the circle the sector is.

Exterior angle of a hexagon = $\frac{360°}{6} = 60°$

Angle inside the sector = $180° - 60° = 120°$

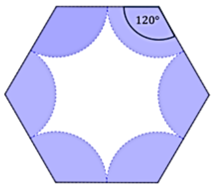
Angle can be measured using a protractor.




↓

The sector is $\frac{120°}{360°} = \frac{1}{3}$ of the circle

Area of sector = $\frac{1}{3}$ of the area of circle




→

$Area\ of\ sector = \frac{1}{3} \times 44.178\ ...$

$Area\ of\ sector = 14.726\ ...\ cm^2$

Shaded area = Area of 6 sectors

Shaded Area = $14.726... \times 6 = 88.36cm^2\ (to\ 2dp)$

## Scenario 6: *In case, a group opts to explore 'Unshaded Area':*

**Chosen Feature: Unshaded Area**

To find the unshaded area, we need to first find the area of the hexagon and then the area of the 6 shaded sectors.

$Unshaded\ area = Area\ of\ hexagon - Shaded\ area$

→

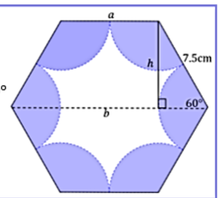
Hypotenuse = 7.5cm *(measured)*

Angle = 60° *(measured)*

Exterior angle of a hexagon = $\frac{360°}{6} = 60°$

Interior angle = $180° - 60° = 120°$

Angle = $\frac{120°}{2} = 60°$



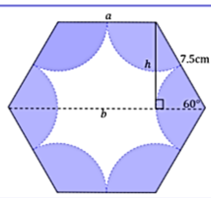

↓

$sin\theta = \frac{opp}{hyp}$

$sin60° = \frac{h}{7.5}$

$h = 7.5sin60°$

$h = 6.495\ ...\ cm$




→

$Area\ of\ trapezium = \frac{1}{2}(7.5 + 15) \times 6.495\ ...$

$Area\ of\ trapezium = 73.070\ ...\ cm^2$

$Area\ of\ hexagon = 73.070\ ... \times 2$

$Area\ of\ hexagon = 146.141\ ...\ cm^2$

↓

$Area\ of\ sector = \frac{1}{3} \times \pi \times 3.75^2$

$Area\ of\ sector = 14.726\ ...\ cm^2$

Shaded area = Area of 6 sectors

Shaded Area = $14.726... \times 6 = 88.357\ ...\ cm^2$

→

$Unshaded\ Area = Area\ of\ hexagon - Shaded\ area$

$Unshaded\ Area = 146.141\ ... - 88.357\ ...$

$Unshaded\ Area = 124.05cm^2\ (to\ 2dp)$

**Scenario 7:** *In case, a group explores 'Trigonometric Ratios or Pythagoras' Theorem':*

**Chosen Feature: Trigonometry/Pythagoras' Theorem**

To find the length of the height of triangle or trapezium, we can use either Pythagoras' Theorem or trigonometric ratios.

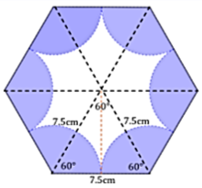
→

Use Pythagoras' Theorem to find a missing side, given the other two sides.

Use trigonometric ratios, to find a missing side, given one side and an angle.

We formed a right-angled triangle to be able to use trigonometric ratios/ Pythagoras' Theorem.

↓





→

Pythagoras' Theorem: $hyp^2 = side^2 + side^2$

$$7.5^2 = h^2 + 3.75^2$$

$$\sqrt{7.5^2 - 3.75^2} = h$$

$$\sqrt{42.1875} = h$$

$$h = 6.495190528cm$$

↓

We can also find the height of the triangle or trapezium using trigonometric ratios:

$$tan\theta = \frac{opp}{adj} \qquad sin\theta = \frac{opp}{hyp}$$

$$tan60° = \frac{h}{3.75} \qquad sin60° = \frac{h}{7.5}$$

$$3.75tan60° = h \qquad 7.5sin60° = h$$

$$h = 6.495190528cm \qquad h = 6.495190528cm$$

→

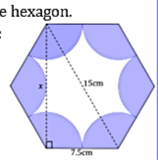
We can also use trigonometric ratios or Pythagoras' theorem to find the distance between two opposite points in the hexagon.

Pythagoras' Theorem: $hyp^2 = side^2 + side^2$

$$15^2 = x^2 + 7.5^2$$

$$\sqrt{15^2 - 7.5^2} = x$$

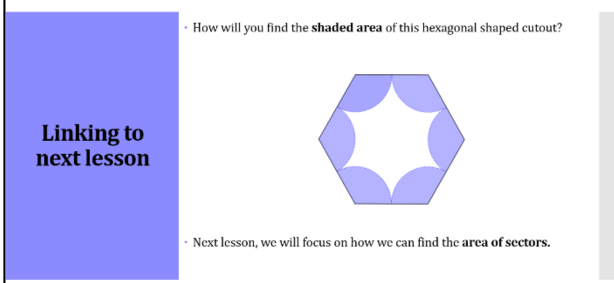
$$x = 12.99cm \ (to \ 2dp)$$




| Closure | |
|---|---|
| **Step 1: Linking to the upcoming lesson** | |
| **Time:** | *2 minutes* |

- The lesson will resume with the teacher assuming the leading role to give an overview of the concepts used in the various investigations conducted by the groups. The teacher will merge the investigations and link the lesson with the learning outcomes of the syllabus.

- Students then will be tasked with exploring the shaded area of the hexagonal-shaped cutout as part of their homework, which will be a pivotal concept in the upcoming lesson.



- In case any group focused on calculating the area of a sector in the first part of the development phase, the teacher will acknowledge their efforts and use their work as a reference to connect it to the upcoming lesson. Their work will be utilised as an example to illustrate how their findings can be expanded upon and refined in the upcoming lesson.

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